Weak quantum hypergroups from finite index C^*-inclusions

(Joint work with: D. Goswami and B. Pal)

Keshab Chandra Bakshi

30 March 2026
Quantum Group Seminar

Regular subfa

$H \leq G$

$\mathcal{N}_G(H)$

$N \subseteq M$

$\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) : uNu^* = N\}$

$\mathcal{N}_M(N)'' = M$. $N \subseteq M \rightarrow$ regular subf

$N \subseteq N \rtimes \mathbb{Z}_2$ $\rightarrow$ regular

$N \subseteq \underset{\overset{\parallel}{M}}{N \rtimes G}$ $\rightarrow$ regular

$N' \cap M = \mathbb{C}$ if the action is 'outer'

'irreducible'

Converse Any irred + reg

$\Rightarrow$ crossed pro by gp

$N \subseteq M$ subfactors II_1-factors

unital

$1_N = 1_M$

Jones' index (1983)

$[M:N]$

$H \leq G$

$x \in M$

$[G:H]$

$\cancel{xNx^* \subseteq N}$

"$LH \subseteq LG$"

$[LG:LH] = \underline{[G:H]}$

Jones index rigidity

$[M:N] \in \{4\cos^2\frac{\pi}{n} : n \geq 3\} \cup [4,\infty]$

$[M:N] = 1 \iff N = M$

$[M:N] = 2 \iff \boxed{M = N \rtimes \mathbb{Z}_2}$

What kind of subfactors can be realized as crossed product by groups irreducible $\qquad$ <u>Regular subfa</u>

'Weak-Hopf alg' (Bohm-Szlachanyi-Nill)

Q: $\qquad$ <u>Jones basic construction</u>

$$\begin{tikzcd}
\underline{\underline{N\subseteq M}} \arrow[r, bend left] \arrow[d] & \underline{\underline{M\subseteq M_1}} \\
R^G\subseteq R \arrow[r, bend left] & R\subseteq R\rtimes G \\
\overset{\text{f.d. } C^*\text{-algs}}{B\subseteq A} \arrow[r] & A\subseteq A_1
\end{tikzcd}$$

$$[M_1 : M] = [M : N]$$

Ocneanu - Szymanski

$\boxed{N \subseteq N \rtimes G \to \text{depth } 2}$ irreducib

$N \subseteq N \rtimes H \to$ depth 2 ..

$\to$ Any irreducible depth 2

$(N \subseteq M)$ $\exists H$ s.t. $N = M^H$

$M = N \rtimes H$

Regular

$\Downarrow$

Depth 2

Nik - Vainerman:

'weak Hopf-alg'

Any depth 2 subf

$N \subseteq N \rtimes WH.$

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

$\longrightarrow$

Jones tower

<u>$[M:N] < \infty$.</u>

$$N' \cap M_k = \{ x \in M_k : xn = nx \ \forall n \in N \}$$

f.d. C^*-alg.

$$\boxed{N' \cap M \subseteq N' \cap M_1 \subseteq N' \cap M_2 \subseteq N' \cap M_3 \subseteq \cdots}$$

<u>Depth 2:</u> $N' \cap M \subseteq N' \cap M_1 \subseteq \underline{N' \cap M_2}$

basic constrn

Can we drop depth 2?

D. Gosw - B-Pal.

$N \subseteq M$ $[M:N] < \infty$.

irreducible: $N' \cap M_1$

Quantum hypergroup (Chapovsky + Vainerman)

Non-irred: weak-Quantum-hyper

$[M:N]$ $N' \cap M_1$

$N \subseteq M$ $B \subseteq A$ C^*-algebra

Preliminary Definitions

Jones index $N \subseteq M$

- For a pair $B \subset A$ of unital C^*-algebras (with common identity), a **conditional expectation** $E : A \to B$ is a positive, norm one projection, which satisfies $E(axb) = aE(x)b$ for all $a, b \in B$ and $x \in A$.

Preliminary Definitions

- For a pair $B \subset A$ of unital C^*-algebras (with common identity), a **conditional expectation** $E : A \to B$ is a positive, norm one projection, which satisfies $E(axb) = aE(x)b$ for all $a, b \in B$ and $x \in A$.
  Example: Consider the map $E : M_2(\mathbb{C}) \to \Delta_2(\mathbb{C})$ defined by

$$E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \underline{\underline{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}}}.$$

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \to B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that
$$x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*, \text{ for every } x \in A.$$

$$E(\lambda_i^* x) = \langle \lambda_i, x \rangle_B$$

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \to B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*$, for every x $\in A$.
- Such a set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a **quasi-basis** for E.

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \rightarrow B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*$, for every x $\in A$.
- Such a set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a **quasi-basis** for E.
- The **Watatani index** of E is given by $\underline{\mathrm{Ind}_w(E)} = \boxed{\sum_{i=1}^{n} \lambda_i \lambda_i^*.}$

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \to B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*$, for every $x \in A$.
- Such a set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a **quasi-basis** for E.
- The **Watatani index** of E is given by $\mathrm{Ind}_w(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*$.
- $\mathrm{Ind}_w(E)$ is not always a scalar, but an invertible positive element of the center of A.

$[M : N]$ $\quad Z(M) = \mathbb{C}.$

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \to B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*$, for every x $\in A$.
- Such a set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a **quasi-basis** for E.
- The **Watatani index** of E is given by $\mathrm{Ind}_w(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*$.
- $\mathrm{Ind}_w(E)$ is not always a scalar, but an invertible positive element of the center of A.
- A C^*-algebra A is called **simple** if it has no non-zero closed ideals.

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani index of a conditional expectation[1]

- Given a pair $B \subset A$ of unital C^*-algebras (with common identity), a conditional expectation $E : A \to B$ is said to have **finite index** if there exist a finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x) = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^*$, for every x $\in A$.
- Such a set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a **quasi-basis** for E.
- The **Watatani index** of E is given by $\mathrm{Ind}_w(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*$.
- $\mathrm{Ind}_w(E)$ is not always a scalar, but an invertible positive element of the center of A.
- A C^*-algebra A is called **simple** if it has no non-zero closed ideals. If $B \subset A$ is a unital inclusion of simple C^*-algebras, then every finite-index conditional expectation has **scalar index**.

---

[1] Y. Watatani, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83** (1990), no. 424, vi+117 pp.

Watatani's C^*-basic construction

- Let $B \subset A$ be an inclusion of unital C^*-algebras and let $E : A \to B$ be a conditional expectation of index-finite type. Then A becomes a **Hilbert B-module** with B-valued inner product $\langle x, y\rangle_B = E(x^*y), \quad x, y \in A$.

Watatani's C^*-basic construction

- Let $B \subset A$ be an inclusion of unital C^*-algebras and let $E : A \to B$ be a conditional expectation of index-finite type. Then A becomes a **Hilbert B-module** with B-valued inner product $\langle x, y\rangle_B = E(x^* y), \quad x, y \in A$.
- The space $\mathcal{L}_B(A)$ of adjointable B-linear maps is a unital C^*-algebra. The algebra A embeds into $\mathcal{L}_B(A)$ via $a \mapsto \lambda(a)$, where $\lambda(a)(x) = ax$ for all $x \in A$.

Watatani's C^*-basic construction

- Let $B \subset A$ be an inclusion of unital C^*-algebras and let $E : A \to B$ be a conditional expectation of index-finite type. Then A becomes a **Hilbert B-module** with B-valued inner product $\langle x, y\rangle_B = E(x^*y), \quad x, y \in A$.
- The space $\mathcal{L}_B(A)$ of adjointable B-linear maps is a unital C^*-algebra. The algebra A embeds into $\mathcal{L}_B(A)$ via $a \mapsto \lambda(a)$, where $\lambda(a)(x) = ax$ for all $x \in A$.
- There exists a projection $e_1 \in \mathcal{L}_B(A)$ (the **Jones projection**) defined by $e_1(x) = E(x), \quad x \in A$.

Watatani's C^*-basic construction

- Let $B \subset A$ be an inclusion of unital C^*-algebras and let $E : A \to B$ be a conditional expectation of index-finite type. Then A becomes a **Hilbert** B-**module** with B-valued inner product $\langle x, y \rangle_B = E(x^* y), \quad x, y \in A$.
- The space $\mathcal{L}_B(A)$ of adjointable B-linear maps is a unital C^*-algebra. The algebra A embeds into $\mathcal{L}_B(A)$ via $a \mapsto \lambda(a)$, where $\lambda(a)(x) = ax$ for all $x \in A$.
- There exists a projection $e_1 \in \mathcal{L}_B(A)$ (the **Jones projection**) defined by $e_1(x) = E(x), \quad x \in A$.
- The C^*-**basic construction** is $A_1 := \overline{\mathrm{span}}\{\underline{\underline{\lambda(x) e_1 \lambda(y)}} : x, y \in A\} \subset \underline{\underline{\mathcal{L}_B(A)}}$.

$\lambda(x)$ $\qquad e_1 \qquad (B \subseteq A, E)$

$\boxed{A_1}$

Watatani's C^*-basic construction

- Let $B \subset A$ be an inclusion of unital C^*-algebras and let $E : A \to B$ be a conditional expectation of index-finite type. Then A becomes a **Hilbert B-module** with B-valued inner product $\langle x, y \rangle_B = E(x^* y), \quad x, y \in A$.
- The space $\mathcal{L}_B(A)$ of adjointable B-linear maps is a unital C^*-algebra. The algebra A embeds into $\mathcal{L}_B(A)$ via $a \mapsto \lambda(a)$, where $\lambda(a)(x) = ax$ for all $x \in A$.
- There exists a projection $e_1 \in \mathcal{L}_B(A)$ (the **Jones projection**) defined by $e_1(x) = E(x), \quad x \in A$.
- The **C^*-basic construction** is $A_1 := \overline{\mathrm{span}}\{\lambda(x) e_1 \lambda(y) : x, y \in A\} \subset \mathcal{L}_B(A)$.
- There exists a dual conditional expectation $E_A^{A_1} : A_1 \to A$ of finite index. Iterating the construction gives

$$\underbrace{B \subset A}_{E_0} \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots .$$

$E_A^{A_1} \to E_1$

e_2

e_k

$(\underline{B} \subseteq A, \underline{\underline{E}})$ $\qquad F : A \to B$

$\text{ind}_w E$ $\qquad \text{ind}_w F.$

$\boxed{\exists!\ E_0}$ which is minimal

$$\text{ind}_w E_0 \leq \text{ind}_w \underline{\underline{E}}$$

$$[A : \underline{B}]_0 = \text{ind}_w E_0$$

$$\underbrace{(\underline{B} \subseteq A, E)} \longrightarrow \boxed{E_0}$$

$$\overset{\tilde{E}_0}{\underbrace{A \subseteq A_1}_{\text{simple}}}$$

Fourier theory on relative commutants

- For an inclusion $B \subset A$ of simple unital C^*-algebras with a conditional expectation of index-finite type, each **relative commutant** $\underline{B' \cap A_n}$ is a **finite-dimensional** C^***-algebra**.

$\{\underline{N' \cap M_k}\}_k$ f. d C^*-algs

M_k - II_1 $\boxed{tr}$

---

[2] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *Operator algebras and applications, Vol. 2*, 119–172, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press.

[3] K. C. Bakshi and V. P. Gupta, Lattice of intermediate subalgebras, *J. Lond. Math. Soc.* (2) **104** (2021).

[4] K. C. Bakshi, S. Guin, B. Pal, and Sruthymurali, Higher reflections and entropy of canonical shifts for inclusions of C^*-algebras with finite Watatani index, Internat. J. Math. **37** (2026).

Fourier theory on relative commutants

- For an inclusion $B \subset A$ of simple unital C^*-algebras with a conditional expectation of index-finite type, each **relative commutant** $B' \cap A_n$ is a **finite-dimensional** C^***-algebra**.
- In subfactor theory, the **Fourier transform** plays a crucial role. **Ocneanu**[2] introduced this map, generalizing the classical Fourier transform for finite abelian groups. This map induces a new multiplication on the second relative commutant, extending the classical **convolution product.**

---

[2] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *Operator algebras and applications, Vol. 2*, 119–172, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press.

[3] K. C. Bakshi and V. P. Gupta, Lattice of intermediate subalgebras, *J. Lond. Math. Soc.* (2) **104** (2021).

[4] K. C. Bakshi, S. Guin, B. Pal, and Sruthymurali, Higher reflections and entropy of canonical shifts for inclusions of $C^\star$-algebras with finite Watatani index, Internat. J. Math. **37** (2026).

Fourier theory on relative commutants

- For an inclusion $B \subset A$ of simple unital C^*-algebras with a conditional expectation of index-finite type, each **relative commutant** $B' \cap A_n$ is a **finite-dimensional** C^***-algebra**.
- In subfactor theory, the **Fourier transform** plays a crucial role. **Ocneanu**[2] introduced this map, generalizing the classical Fourier transform for finite abelian groups. This map induces a new multiplication on the second relative commutant, extending the classical **convolution product**.
- Analogous to the subfactor setting, the first author and Ved Gupta [3] developed a Fourier theory in the C^*-algebra setting.

---

[2] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *Operator algebras and applications, Vol. 2*, 119–172, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press.

[3] K. C. Bakshi and V. P. Gupta, Lattice of intermediate subalgebras, *J. Lond. Math. Soc.* (2) **104** (2021).

[4] K. C. Bakshi, S. Guin, B. Pal, and Sruthymurali, Higher reflections and entropy of canonical shifts for inclusions of $C^\star$-algebras with finite Watatani index, Internat. J. Math. **37** (2026).

Fourier theory on relative commutants

- For an inclusion $B \subset A$ of simple unital C^*-algebras with a conditional expectation of index-finite type, each **relative commutant** $B' \cap A_n$ is a **finite-dimensional C^*-algebra**.
- In subfactor theory, the **Fourier transform** plays a crucial role. **Ocneanu**[2] introduced this map, generalizing the classical Fourier transform for finite abelian groups. This map induces a new multiplication on the second relative commutant, extending the classical **convolution product**.
- Analogous to the subfactor setting, the first author and Ved Gupta [3] developed a Fourier theory in the C^*-algebra setting. Using the Fourier transform, **reflection operators**[4] $r_n^+ : B' \cap A_n \to B' \cap A_n$ were introduced recently. These operators have important applications and will play an important role in what follows.

---

[2] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *Operator algebras and applications, Vol. 2*, 119–172, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press.

[3] K. C. Bakshi and V. P. Gupta, Lattice of intermediate subalgebras, *J. Lond. Math. Soc.* (2) **104** (2021).

[4] K. C. Bakshi, S. Guin, B. Pal, and Sruthymurali, Higher reflections and entropy of canonical shifts for inclusions of $C^\star$-algebras with finite Watatani index, Internat. J. Math. **37** (2026).

Fourier theory contd.

Using the minimal conditional expectations, one obtains "Markov-type traces" on these algebras. Specifically, for $n \geq 0$, the map

$$✓\ \mathrm{tr}_n := (E_0 \circ E_1 \circ \cdots \circ E_n)\big|_{B' \cap A_n} : B' \cap A_n \to \mathbb{C}$$

$$B' \cap A_{n-1} \subseteq \boxed{B' \cap A_n}$$

defines a faithful tracial state on $B' \cap A_n$.

Theorem (Gup- JLMS)

For all $n \geq 1$, $\mathrm{tr}_n(xe_n) = \tau\, \mathrm{tr}_{n-1}(x)$ *for every* $x \in B' \cap A_{n-1}$, *and* $\mathrm{tr}_n|_{B' \cap A_{n-1}} = \mathrm{tr}_{n-1}$.

" $[A:B]_0^{-1}$

When there is no risk of confusion, we shall drop the subscript and write simply tr for notational convenience. If $\{\lambda_i : 1 \leq i \leq n\} \subset A$ is a quasi-basis for the minimal conditional expectation E_0. Then the tr-preserving conditional expectation from $B' \cap A_n$ onto $A' \cap A_n$ is given by the following:

$$✓\ E^{B' \cap A_n}_{A' \cap A_n}(x) = \tau \sum_i \lambda_i x \lambda_i^* \qquad \text{for all } x \in B' \cap A_n.$$

$(\underline{\underline{B \subseteq A}}, E)$ $\quad \exists\, E_0$ minimal conditional expectation

$(\underline{\underline{A \subseteq A_1}}, E_1)$ $\quad E_1$ mini

$(B \subseteq A_1, E_1 \circ E_0)$ $\quad E_1 : A_1 \to A$.

$E_0 : A \to B$

minimal. $\quad E_1 \circ E_0 : A_1 \to B$

$\underbrace{B \subseteq \overbrace{C}^{} \subseteq A}$ $\quad [A : B]_0 = [A : C]_0\, [C : B]_0$

~~$B \subseteq A$~~

~~$A' \subseteq B$~~

$$\begin{tikzcd}
B' \cap A_0 \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\vee"] & B' \cap A_1 \arrow[r, phantom, "\subseteq", ", tn"] \arrow[d, phantom, "\cup"] & B' \cap A_2 \arrow[r, phantom, "\subseteq", ", tn_2"] \arrow[d, phantom, "\cup"] \arrow[d, bend left=60] & \cdots \\
A' \cap A \arrow[r, phantom, "\subseteq"] & A' \cap A_1 \arrow[r, phantom, "\subseteq"] & A' \cap A_2 \arrow[r, phantom, "\subseteq"] & \cdots
\end{tikzcd}$$

$$B' \cap A_n \longrightarrow A' \cap A_n$$

Fourier theory contd.

$G \longrightarrow \hat{G}$

$\hat{B} \subseteq B.$ $\quad \mathcal{F}: B' \cap A_1 \to A' \cap A_2$

For each $n \geq 0$, the Fourier transform $\mathcal{F}_n : \underline{B' \cap A_n} \to \underline{A' \cap A_{n+1}}$ is defined by

$$✓\ \mathcal{F}_n(x) = \tau^{-\frac{n+2}{2}} E^{B' \cap A_{n+1}}_{A' \cap A_{n+1}}(x v_{n+1}), \qquad x \in B' \cap A_n,$$

where $v_{n+1} = \underline{e_{n+1} \cdots e_1}$. The inverse Fourier transform $\mathcal{F}_n^{-1} : A' \cap A_{n+1} \to B' \cap A_n$ is given by

$$✓\ \mathcal{F}_n^{-1}(w) = \tau^{-\frac{n+2}{2}} E_{n+1}(w v_{n+1}^*), \qquad w \in A' \cap A_{n+1}.$$

This terminology is justified because, for every $n \geq 0$,
$\underline{\mathcal{F}_n \circ \mathcal{F}_n^{-1} = \mathrm{id}_{A' \cap A_{n+1}}}$ and $\underline{\mathcal{F}_n^{-1} \circ \mathcal{F}_n = \mathrm{id}_{B' \cap A_n}}$,

Fourier theory contd.

$\underline{\underline{x \star y}}$. x, y $B' \cap A_1$

For $x, y \in B' \cap A_n$, the convolution product is defined by

$$\underline{\underline{x \star y := \mathcal{F}_n^{-1}\big(\mathcal{F}_n(y)\,\mathcal{F}_n(x)\big)}}, \tag{0.1}$$

which is associative. Furthermore, if $x, y \in \underline{\underline{B'}} \cap A_1$,

$$\underline{\underline{(x \star y)^* = x^* \star y^*}}, \tag{0.2}$$

as stated. However, this $*$-compatibility does not generally extend to higher relative commutants.

$B' \cap A_n$

Fourier theory contd.

$\rho_n^{n+1} = id.$ $\rho_n^2 = id.$

For each $n \geq 0$, the rotation operator $\rho_n^+ : B' \cap A_n \to B' \cap A_n$ is defined by

$$\rho_n^+(x) = \left(\mathcal{F}_n^{-1}(\mathcal{F}_n(x)^*)\right)^*, \qquad x \in B' \cap A_n.$$

Reflection operators provide powerful tools for establishing Fourier-theoretic inequalities on the higher relative commutants and for connecting the Connes-Størmer entropy of the canonical shift with the minimal Watatani index.
For every $n \geq 0$, $r_{2n+1}^+ := (\rho_{2n+1}^+)^{n+1}$, defines a map $r_{2n+1}^+ : B' \cap A_{2n+1} \to B' \cap A_{2n+1}$. For $n = 1$, these operators admit the following concrete expression that will be used later. For all $x \in B' \cap A_1$,

$$r_1^+(x) = \tau^{-1} \sum_i E_1(e_1 \lambda_i x)\, e_1 \lambda_i^* . \tag{0.3}$$

$B' \cap A_1$

$r_1^2 = id.$

Fourier theory contd.

$B \subseteq A$ $\quad A_1 \subseteq A_2$ $\quad B \subseteq A$ $\quad A \subseteq A_1$ $\quad A_1 \subseteq A$

$B' \cap A_1 \longleftrightarrow A_1' \cap A_3$ $\quad B' \cap A_n, tr_n$

The reflection operators enjoy the following structural properties:

Theorem (___, Gupta) + B-Pal, S. Gu, Srutameerah

For every $n \geq 0$, the reflection operator r^+_{2n+1} is a unital, involutive, $*$-preserving anti-homomorphism, and moreover it preserves the trace.

In the setting of simple unital C^*-algebras, one also defines the shift operators. For each $n \geq 0$, $S^+_n := r^+_{2n+3} \circ r^+_{2n+1}$. Each map $S^+_n : B' \cap A_{2n+1} \to A'_1 \cap A_{2n+3}$ is a unital, trace-preserving $*$-isomorphism. The reflection operator S^+_0 admit the following explicit form:

$$S^+_0(x) = \tau^{-2} \sum_i \lambda_i e_1 e_2 y e_3 e_2 e_1 \lambda_i^*, \qquad \text{for } x \in B' \cap A_1. \tag{0.4}$$

What is a Hopf Algebra?

Definition. A *Hopf algebra* over a field k is a vector space H equipped with:

- **Multiplication** $m : H \otimes H \to H$, **Unit** $u : k \to H$
- **Comultiplication** $\Delta : H \to H \otimes H$, **Counit** $\varepsilon : H \to k$
- **Antipode** $S : H \to H$

Compatibility:

- Δ and ε are algebra homomorphisms
- $m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = u\varepsilon$

Example: Group algebra $k[G]$

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1$$

Weak C^*-Hopf algebra

- A **weak bialgebra** is a quintuple $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ where $(\mathcal{P}, m, \eta)$ is an algebra and $(\mathcal{P}, \Delta, \varepsilon)$ is a coalgebra, satisfying the following compatibility conditions:
  - *(i)* Δ is an algebra homomorphism,
  - *(ii)* $\varepsilon(yy'y'') = \varepsilon(yy'_{(1)})\,\varepsilon(y'_{(2)}y'') = \varepsilon(yy'_{(2)})\,\varepsilon(y'_{(1)}y'')$, for all $y, y', y'' \in \mathcal{P}$,
  - *(iii)* The element $\Delta(1) \in \mathcal{P} \otimes \mathcal{P}$ satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta(1) = (\Delta(1) \otimes 1)\big(1 \otimes \Delta(1)\big) = \big(1 \otimes \Delta(1)\big)\big(\Delta(1) \otimes 1\big).$$

$\Delta(1) \neq 1.$

Weak C^*-Hopf algebra

- A **weak bialgebra** is a quintuple $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ where $(\mathcal{P}, m, \eta)$ is an algebra and $(\mathcal{P}, \Delta, \varepsilon)$ is a coalgebra, satisfying the following compatibility conditions:
  - (i) Δ is an algebra homomorphism,
  - (ii) $\varepsilon(yy'y'') = \varepsilon(yy'_{(1)})\,\varepsilon(y'_{(2)}y'') = \varepsilon(yy'_{(2)})\,\varepsilon(y'_{(1)}y'')$, for all $y, y', y'' \in \mathcal{P}$,
  - (iii) The element $\Delta(1) \in \mathcal{P} \otimes \mathcal{P}$ satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta(1) = (\Delta(1) \otimes 1)\big(1 \otimes \Delta(1)\big) = \big(1 \otimes \Delta(1)\big)\big(\Delta(1) \otimes 1\big).$$

- A **weak Hopf algebra** is a weak bialgebra $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ endowed with a linear map $S : \mathcal{P} \to \mathcal{P}$, referred to as the **antipode**, which satisfies the following relations for all $y \in \mathcal{P}$,
  - (i) $y_{(1)}S(y_{(2)}) = \varepsilon(1_{(1)}y)1_{(2)}$,
  - (ii) $S(y_{(1)})y_{(2)} = 1_{(1)}\varepsilon(y1_{(2)})$,
  - (iii) $S(y_{(1)})y_{(2)}S(y_{(3)}) = S(y)$.

Weak C^*-Hopf algebra

- A **weak bialgebra** is a quintuple $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ where $(\mathcal{P}, m, \eta)$ is an algebra and $(\mathcal{P}, \Delta, \varepsilon)$ is a coalgebra, satisfying the following compatibility conditions:
  - (i) Δ is an algebra homomorphism,
  - (ii) $\varepsilon(yy'y'') = \varepsilon(yy'_{(1)})\,\varepsilon(y'_{(2)}y'') = \varepsilon(yy'_{(2)})\,\varepsilon(y'_{(1)}y'')$, for all $y, y', y'' \in \mathcal{P}$,
  - (iii) The element $\Delta(1) \in \mathcal{P} \otimes \mathcal{P}$ satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta(1) = (\Delta(1) \otimes 1)\big(1 \otimes \Delta(1)\big) = \big(1 \otimes \Delta(1)\big)\big(\Delta(1) \otimes 1\big).$$

- A **weak Hopf algebra** is a weak bialgebra $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ endowed with a linear map $S : \mathcal{P} \to \mathcal{P}$, referred to as the **antipode**, which satisfies the following relations for all $y \in \mathcal{P}$,
  - (i) $y_{(1)}S(y_{(2)}) = \varepsilon(1_{(1)}y)1_{(2)}$,
  - (ii) $S(y_{(1)})y_{(2)} = 1_{(1)}\varepsilon(y1_{(2)})$,
  - (iii) $S(y_{(1)})y_{(2)}S(y_{(3)}) = S(y)$.
- A **weak C^*-Hopf algebra** is a weak Hopf algebra $(\mathcal{P}, m, \eta, \Delta, \varepsilon)$ for which $\mathcal{P}$ is a finite-dimensional C^*-algebra and the comultiplication Δ is $*$-preserving.

Symmetries for depth 2 inclusions[5]

- An inclusion $B \subset A$ of unital C^*-algebras with a conditional expectation of index-finite type is said to have **finite depth** if there exists an integer $n \geq 1$ satisfying $\underline{(B' \cap A_{n-1})\, e_n \,(B' \cap A_{n-1}) = B' \cap A_n}$. The least such integer n is termed as the **depth of the inclusion**.

$$B' \cap A \subsetneq B' \cap A_1 \subsetneq B' \cap A_2$$

$C_2 \nearrow$

<u>depth 2</u>

---

[5] B. Pal, Depth 2 inclusions of simple C^*-algebras and their weak C^*-Hopf algebra symmetries, arXiv:2511.19954.

Symmetries for depth 2 inclusions[5]

- An inclusion $B \subset A$ of unital C^*-algebras with a conditional expectation of index-finite type is said to have **finite depth** if there exists an integer $n \geq 1$ satisfying $(B' \cap A_{n-1})\, e_n\, (B' \cap A_{n-1}) = B' \cap A_n$. The least such integer n is termed as the **depth of the inclusion**.

Theorem

Let $B \subset A$ be a depth 2 inclusion of simple unital C^*-algebras. Then the second relative commutant $B' \cap A_1$ admits a ***weak*** *C^*-**Hopf algebra** structure.*

$M = N \rtimes WH$

$N = M^{WH}$

$N \subseteq M$

$$\begin{tikzcd}
N' \cap M_1 \arrow[r, Leftrightarrow] \arrow[d] & M' \cap M_2 \\
WH.
\end{tikzcd}$$

---

[5] B. Pal, Depth 2 inclusions of simple C^*-algebras and their weak C^*-Hopf algebra symmetries, arXiv:2511.19954.

Symmetries for depth 2 inclusions[5]

- An inclusion $B \subset A$ of unital C^*-algebras with a conditional expectation of index-finite type is said to have **finite depth** if there exists an integer $n \geq 1$ satisfying $(B' \cap A_{n-1})\, e_n\, (B' \cap A_{n-1}) = B' \cap A_n$. The least such integer n is termed as the **depth of the inclusion**.

Theorem

Let $B \subset A$ be a depth 2 inclusion of simple unital C^*-algebras. Then the second relative commutant $B' \cap A_1$ admits a ***weak C^*-Hopf algebra*** *structure.*

Theorem

Let $B \subset A$ be as above. Then weak C^*-Hopf algebra $B' \cap A_1$ acts on A such that B is the ***fixed-point subalgebra*** *and $A_1 \cong A \rtimes (B' \cap A_1)$.*

---

[5] B. Pal, Depth 2 inclusions of simple C^*-algebras and their weak C^*-Hopf algebra symmetries, arXiv:2511.19954.

Quantum hypergroup

Definition (Chapovsky and Vainerman)

A quadruple $(A, \Delta, \varepsilon, \star)$ *is said to define a* ***quantum hypergroup*** *structure on the* C^**-algebra* $(A, \cdot, 1, *)$ *if the following conditions are satisfied:*

- *The system* $(A, \Delta, \varepsilon, \star)$ *forms a* $*$*-coalgebra with counit* ε*; that is,* $\Delta\colon A \to A \otimes A$ *and* $\varepsilon\colon A \to \mathbb{C}$ *are linear maps, and* $\star\colon A \to A$ *is an antilinear map such that*
  - *(i)* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$
  - *(ii)* $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id},$
  - *(iii)* $\Delta \circ \star = \Pi \circ (\star \otimes \star) \circ \Delta,$
  - *(iv)* $\star \circ \star = \mathrm{id}$, *where* $\Pi\colon A \otimes A \to A \otimes A$ *denotes the flip map.*

Quantum hypergroup

Definition (Chapovsky and Vainerman)

(1999) 'JGT'

A quadruple $(A, \Delta, \varepsilon, \star)$ *is said to define a* ***quantum hypergroup*** *structure on the* C^**-algebra* $(A, \cdot, 1, *)$ *if the following conditions are satisfied:*

- *The system* $(A, \Delta, \varepsilon, \star)$ *forms a* $*$*-coalgebra with counit* ε*; that is,* $\Delta\colon A \to A \otimes A$ *and* $\varepsilon\colon A \to \mathbb{C}$ *are linear maps, and* $\star\colon A \to A$ *is an antilinear map such that*
  - (*i*) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
  - (*ii*) $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$,
  - (*iii*) $\Delta \circ \star = \Pi \circ (\star \otimes \star) \circ \Delta$,
  - (*iv*) $\star \circ \star = \mathrm{id}$, *where* $\Pi\colon A \otimes A \to A \otimes A$ *denotes the flip map.*
- *The comultiplication* $\Delta\colon A \to A \otimes A$ *is positive.*
- *The following compatibility conditions are satisfied for all* $a, b \in A$*:*
  - (*i*) $(a \cdot b)^\star = a^\star \cdot b^\star$,
  - (*ii*) $\Delta \circ * = (* \otimes *) \circ \Delta$,
  - (*iii*) $\varepsilon(a \cdot b) = \varepsilon(a)\varepsilon(b)$,
  - (*iv*) $\Delta(1) = 1 \otimes 1$,
  - (*v*) $\star \circ * = * \circ \star$.

Weak quantum hypergroup

- Thus a **quantum hypergroup**[6] is a unital C^*-algebra equipped with a completely positive, coassociative coproduct.

---

[6] Y. A. Chapovsky and L. I. Vainerman, Compact quantum hypergroups, J. Operator Theory **41** (1999).

Weak quantum hypergroup

- Thus a **quantum hypergroup**[6] is a unital C^*-algebra equipped with a completely positive, coassociative coproduct.

Definition (___, Goswami, Pal)

Let $(A, \cdot, 1, *)$ *be a unital* C^**-algebra. We say that* $(A, \Delta, \varepsilon, \#)$ *defines a* ***weak quantum hypergroup*** *structure on* $(A, \cdot, 1, *)$ *if the following conditions hold:*

- (A, Δ, ε) *is a coalgebra and* $\# : A \to A$ *is an antilinear map satisfying:*
  - (i) $\Delta \circ \# = \Pi \circ (\# \otimes \#) \circ \Delta$, *where* $\Pi : A \otimes A \to A \otimes A$ *is the flip map,*
  - (ii) $\# \circ \# = \mathrm{id}$.
- *The map* Δ *is positive.*
- *The following identities hold:*
  - (i) $(x \cdot y)^\# = x^\# \cdot y^\#$, *for all* $x, y \in A$,
  - (ii) $\Delta \circ * = (* \otimes *) \circ \Delta$,
  - (iii) $\# \circ * = * \circ \#$.
  - (iv) $\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z)$, *for all* $x, y, z \in A$.
  - (v) $(\mathrm{id} \otimes \Delta) \circ \Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$.

$\Delta(1) = \underline{1 \otimes 1},$

---

[6] Y. A. Chapovsky and L. I. Vainerman, Compact quantum hypergroups, J. Operator Theory **41** (1999).

Symmetries for arbitrary inclusions of simple C^*-algebras[7]

On $B' \cap A_1$, we define the maps Δ, ε, and # as follows:

- $\langle \Delta(x), y \otimes z \rangle = \tau^{\frac{1}{2}} \langle x, y \star z \rangle$,
- $\varepsilon(x) = \tau^{-1} \langle e_1, x \rangle$,
- $x^\# = r_1^+(x^*)$, for any $x \in B' \cap A_1$.

---

[7] K.C. Bakshi, D. Goswami, and B. Pal, Weak quantum hypergroups from finite index C*-inclusions, arXiv:2601.12406.

Symmetries for arbitrary inclusions of simple C^*-algebras[7]

On $B' \cap A_1$, we define the maps Δ, ε, and # as follows:

- $\langle \Delta(x), y \otimes z \rangle = \tau^{\frac{1}{2}} \langle x, y \star z \rangle$,
- $\varepsilon(x) = \tau^{-1} \langle e_1, x \rangle$,
- $x^{\#} = r_1^{+}(x^*)$, for any $x \in B' \cap A_1$.

$B \subseteq A$

Theorem (___, Goswami, Pal)

Let $B \subset A$ be an inclusion of simple unital C^*-algebras with a conditional expectation of index-finite type. Then we have the following results:

- $(B' \cap A_1, \Delta, \varepsilon, \#)$ *becomes a* ***weak quantum hypergroup***.

---

[7] K.C. Bakshi, D. Goswami, and B. Pal, Weak quantum hypergroups from finite index C*-inclusions, arXiv:2601.12406.

Symmetries for arbitrary inclusions of simple C^*-algebras[7]

On $B' \cap A_1$, we define the maps Δ, ε, and # as follows:

- $\big\langle \Delta(x), y \otimes z \big\rangle = \tau^{\frac{1}{2}} \big\langle x, y \star z \big\rangle$,
- $\varepsilon(x) = \tau^{-1} \big\langle e_1, x \big\rangle$,
- $x^{\#} = r_1^{+}(x^*)$,   for any $x \in B' \cap A_1$.

Theorem (___, Goswami, Pal)

Let $B \subset A$ be an inclusion of simple unital C^*-algebras with a conditional expectation of index-finite type. Then we have the following results:

- $(B' \cap A_1, \Delta, \varepsilon, \#)$ *becomes a* ***weak quantum hypergroup****.*
- *If the inclusion is irreducible, then the weak quantum hypergroup structure on $B' \cap A_1$ upgrades to that of a* ***(genuine) quantum hypergroup****.*

---

[7] K.C. Bakshi, D. Goswami, and B. Pal, Weak quantum hypergroups from finite index C*-inclusions, arXiv:2601.12406.

Coalgebra structure on the second relative commutant[8]

Definition (___, Goswami, Pal)

On the second relative commutant $B' \cap A_1$*, we define the maps* Δ *and* ε *as follows:*

- $\Delta : B' \cap A_1 \to (B' \cap A_1) \otimes (B' \cap A_1) : \quad \langle \Delta(x), y \otimes z \rangle = \tau^{\frac{1}{2}} \langle x, y \star z \rangle,$
- $\varepsilon : B' \cap A_1 \to \mathbb{C} : \quad \varepsilon(x) = \tau^{-1} \langle e_1, x \rangle,$

for all $x, y, z \in B' \cap A_1$*. Here the inner product on* $B' \cap A_1$ *is given by* $\langle x, y \rangle = \operatorname{tr}(x^* y)$*.*

$x, y \in B' \cap A_1$ $\qquad$ $\langle x, y \rangle = \operatorname{tr}(x^* y)$

---

[8] K.C. Bakshi, D. Goswami, and B. Pal, Weak quantum hypergroups from finite index C*-inclusions, arXiv:2601.12406.

Coalgebra structure on the second relative commutant[8]

$(x \star y) \star z = x \star (y \star z)$ ✓

Definition (___, Goswami, Pal)

On the second relative commutant $B' \cap A_1$, we define the maps Δ and ε as follows:

- $\Delta : B' \cap A_1 \to (B' \cap A_1) \otimes (B' \cap A_1) : \quad \langle \Delta(x), y \otimes z \rangle = \tau^{\frac{1}{2}} \langle x, y \star z \rangle,$
- $\varepsilon : B' \cap A_1 \to \mathbb{C} : \quad \varepsilon(x) = \tau^{-1} \langle e_1, x \rangle,$

for all $x, y, z \in B' \cap A_1$. Here the inner product on $B' \cap A_1$ is given by $\langle x, y \rangle = \mathrm{tr}(x^* y)$.

Proposition (___, Goswami, Pal)

$(B' \cap A_1, \Delta, \varepsilon)$ *becomes a* ***coalgebra*** *and coproduct map Δ is* ***completely positive****.*

---

[8] K.C. Bakshi, D. Goswami, and B. Pal, Weak quantum hypergroups from finite index C*-inclusions, arXiv:2601.12406.

Sketch of the proof

$B\subseteq A$ S $A_1\subseteq A_2$

$B'\cap A_1$ $A_1'\cap A_3$

T

Theorem

The C^*-algebras $(B'\cap A_1)\otimes(B'\cap A_1)$ and $\underline{(B'\cap A_1)(A_1'\cap A_3)}$ are isomorphic.

Define $T:\underline{(B'\cap A_1)\otimes(B'\cap A_1)}\to\underline{(B'\cap A_1)(A_1'\cap A_3)}$ by

$$T(x\otimes y)=x\,\underline{S_0^+}(y),\qquad x,y\in B'\cap A_1.$$

where S_0^+ is the shift operator on $B'\cap A_1$ as defined. Then clearly T is an $*$-isomorphism.

Sketch contd....

$$E^{\underline{B'\cap A_3}}_{\underline{(B'\cap A_1)(A_1'\cap A_3)}}$$

Theorem

For $z \in B' \cap A_1$, we have

$$\underline{\Delta(z)} = \cancel{\tau^{-2}}\, T^{-1} \circ E(\underline{e_2' z e_3' e_2'}),$$

where E is the trace preserving conditional expectation from $B' \cap A_3$ onto $(B' \cap A_1)(A_1' \cap A_3)$.

It is enough to prove that for any $x, y \in B' \cap A_1$,

$$\tau^{-2} \langle T^{-1} \circ E(e_2 z e_3 e_2), x \otimes y \rangle = \tau^{\frac{1}{2}} \langle z, x \star y \rangle.$$

Sketch contd....

Δ

irreducible

$\Delta(1) = 1 \otimes 1.$

π_1 τ_1

Theorem

Δ is a $*$-preserving completely positive map.

We have

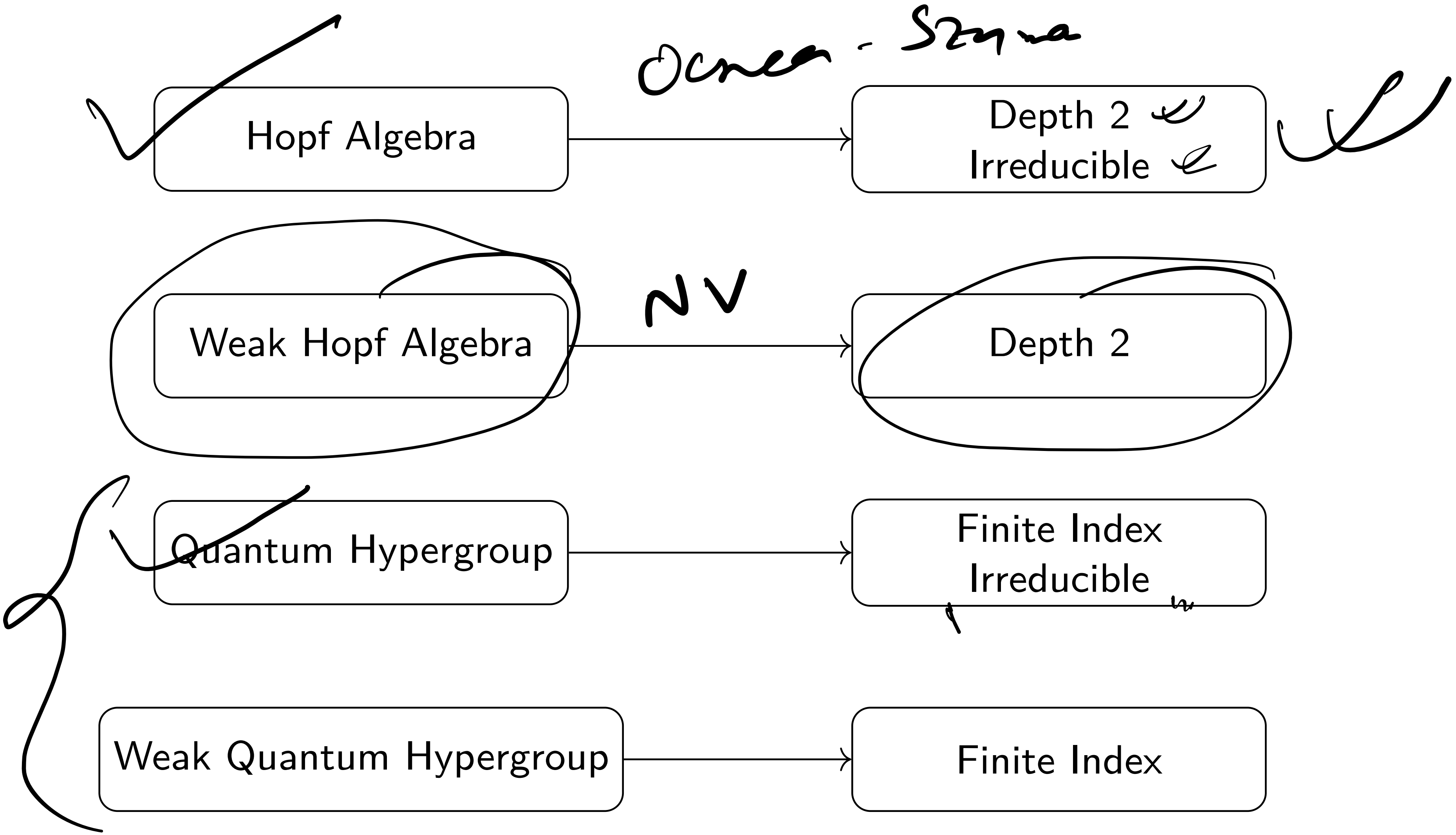
$$\Delta(z) = \tau^{-1} T^{-1} \circ E(\tau^{-1} e_2 z e_3 e_2).$$

It is easy to verify that the map $z \to \tau^{-1} e_2 z e_3 e_2$ is a $*$-algebra homomorphism. Hence, Δ is $*$-preserving. Since any $*$-algebra homomorphism between two C^*-algebras is completely positive, it follows that the map $z \to \tau^{-1} e_2 z e_3 e_2$ is completely positive. Moreover, since T^{-1} is a $*$-isomorphism, it is also completely positive. Therefore, as a composition of completely positive maps, Δ is completely positive.

$B' \cap A_1$ → weak QH.

$A' \cap A_2$ weak QH

Quantum Symmetries and Subfactors



Ocnea - Szyma

Hopf Algebra → Depth 2, Irreducible

NV

Weak Hopf Algebra → Depth 2

Quantum Hypergroup → Finite Index, Irreducible

Weak Quantum Hypergroup → Finite Index

Philosophy: Weakening symmetry ⇒ more general subfactors

Thank you.